# How Many Digits To Include

Here are some simple rules that apply whenever you are writing down a number:

[1](http://www.av8n.com/physics/uncertainty.htm" \l "intro-many) Use many enough digits to avoid unintended loss of information.

[2](http://www.av8n.com/physics/uncertainty.htm" \l "intro-few). Use few enough digits to be reasonably convenient.

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| **Important note:** The previous two sentences tell you everything you need to know for most purposes, including real-life situations as well as academic situations at every level from primary school up to and including introductory college level. |

[3](http://www.av8n.com/physics/uncertainty.htm#intro-calculator-store). When using a calculator, it is good practice to leave intermediate results in the machine. This is simultaneously more accurate and more convenient than writing them down and then keying them in again.

Seriously: The primary rule is to use plenty of digits. You hardly even need to think about it. Too many is vastly better than too few. To say the same thing the other way: If you ever have more digits than you need *and* they are causing major inconvenience, then you can think about reducing the number of digits.

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# When to Write Down Uncertainty

In many cases, when you write down a number, you need not *and should not* associate it with any notion of uncertainty.

* One way this can happen is if you have a number with zero uncertainty. If you roll a pair of dice and observe five spots, the number of spots is 5. This is a raw data point, with no uncertainty whatsoever. So just write down the number. Similarly, the number of centimeters per inch is 2.54, by definition, with no uncertainty whatsoever. Again: just write down the number.
* Another possibility is that there is a cooked data blob, which in principle must have “some” uncertainty, but the uncertainty is too small to be interesting. It is insignificant. It is unimportant. It is immaterial. There are plenty of situations a moderately rough approximation is sufficient. There are even some situations where an *extremely* rough approximation is called for, as in so-called “Fermi” problems.

Along the same lines, here is a less-extreme example that arises in the introductory chemistry class. Suppose the assignment is to balance the equation for the combustion of gasoline, namely

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| |  |  |  |  |  | | --- | --- | --- | --- | --- | | *a* C8H18 + *b* O2 |  | → |  | *x* CO2 + *y* H2O | | ([1](http://www.av8n.com/physics/uncertainty.htm#eq-gasoline)) |

by finding numerical values for the coefficients *a*, *b*, *x*, and *y*. The conventional answer is (*a*, *b*, *x*, *y*) = (2, 25, 16, 18). The outcome of the real reaction must have “some” uncertainty, because there will generally be some nonidealities, including the presence of other molecules such as CO or C60, not to mention NO2 or whatever. However, my point is that we don’t necessarily care about these nonidealities. We can perfectly well find the idealized solution to the idealized equation and postpone worrying about the nonidealities and uncertainties until much, much later.

As another example, suppose you use a digital stopwatch to measure some event, and the reading is 1.234 seconds. We call this number the *indicated* time, and we distinguish it from the *true* time of the event. In principle, there is no chance that the indicated time will be exactly equal to the true time (since true time is a continuous variable, whereas the indicated time is quantized). However, in many cases you may decide that it is close enough, in which case you should just write down the indicated reading and not worry about the quantization error.

Let us continue with the stopwatch example. Suppose we make two observations. The first reading is 1.234 seconds, and the second reading is just the same, namely 1.234 seconds. Meanwhile, however, you may believe that if you repeated the experiment many times, the resulting set of readings would have some amount of scatter, namely 0.01 seconds. The two observations that we actually have don’t show any scatter at all, so your estimate of the uncertainty remains hypothetical and theoretical. Theoretical information is still information, and should be written down in the lab book, plain and simple. For example, you might write a sentence that says “Intuition suggests the timing data is reproducible 0.01 seconds.” It would be even better to include some explanation of why you think so. The principle is simple: Write down what you know. Say what you mean, and mean what you say. The same principle applies to the indicated values. The recommend practice is to write down each indicated value, as-is, plain and simple.

You are not trying write down the true values. You don’t know the true values (except insofar as the indicated values represent them, indirectly). You don’t need to know the true values, so don’t worry about it. The rule is: *Write down what you know.* So write down the indicated value. Also: You are not obliged to attribute any uncertainty to the numbers you write down. Normal lab-book entries do not express an uncertainty using *AB* notation or otherwise, and they do not “imply” an uncertainty using sig figs or otherwise. We are always uncertain about the true value, but we aren’t writing down the true value, so that’s not a concern.

Some people say there must be some uncertainty “associated” with the number you write down, and of course there is, indirectly, in the sense that the indicated value is “associated” with some range of true values. We are always uncertain about the true value, but that does not mean we are uncertain about the indicated value. These things are “associated” ... but they are not the same thing.

In a well-designed experiment, things like readability and quantization error usually do not make a large contribution to the overall uncertainty anyway. Please do not confuse such things with “the” uncertainty.

It is usually a good practice to keep all the original data. When reading an instrument, read it as precisely as the instrument permits, and write down the reading “as is” ... without any conversions, any roundoff, or anything else.

# Why We’re Not Using Significant Figures

No matter what you are trying to do, significant figures are the wrong way to do it.

When writing, do not use the number of digits to imply anything about the uncertainty. If you want to describe a distribution, describe it explicitly, perhaps using expressions such as 1.2340.055.

When reading, do not assume the number of digits tells you anything about the overall uncertainty, accuracy, precision, tolerance, or anything else, unless you are absolutely sure that’s what the writer intended ... and even then, beware that the meaning is very unclear.

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| |  |  | | --- | --- | | |  | | --- | | People who care about their data don’t use sig figs. | | |  |

Significant-digit dogma destroys your data and messes up your thinking in many ways, including:

1. Given a distribution that can be described by an expression such as *AB*, such as 1.2340.055, converting it to sig figs gives you an excessively crude and erratic representation of the uncertainty, *B*.
2. Converting to sig figs also causes excessive roundoff error in the nominal value, *A*. This is a big problem.
3. Sig figs cause people to misunderstand the distinction between roundoff error and uncertainty.
4. Sig figs cause people to misunderstand the distinction between uncertainty and significance. Sig figs cause people to misunderstand the distinction between the *indicated value* and the corresponding range of *true values*.
5. Sig figs cause people to misunderstand the distinction between distributions and numbers. Distributions have width, whereas numbers don’t. Uncertainty is necessarily associated with some distribution, not with any particular point that might have been drawn from the distribution.
6. As a consequence, sig figs make people hesitate to write down numbers. They think they need to know the amount of supposedly “associated” uncertainty before they can write the number, when in fact they don’t. Very commonly, there simply isn’t any “associated” uncertainty anyway.
7. Sig figs weaken people’s understanding of the axioms of the decimal numeral system.
8. Sig figs provide no guidance as to the appropriate decimal representation for repeating decimals such as 80 ÷ 81, or irrational numbers such as √2 or π.

# Crank Three Times™

Here’s a simple yet powerful way of estimating the uncertainty of a result, given the uncertainty of the thing(s) it depends on.

Here’s the procedure, in the simple case when there is only one input variable with appreciable uncertainty:

1. Set up the calculation. Do it once in the usual way, using the nominal, best-estimate values for all the input variables.
2. Then re-do the calculation with the uncertain variable at the end of its upper error bar.
3. Then re-do the calculation with the uncertain variable at the end of its lower error bar.

I call this the *Crank Three Times*™ method. Here is an example:

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|  | |  |  |  |  |  |  | | --- | --- | --- | --- | --- | --- | | *x* |  |  |  |  | 1/*x* | | === |  |  |  |  | === | | 2.02 | (high case) |  | → |  | .495 | | 2 | (nominal case) |  | → |  | .5 | | 1.98 | (low case) |  | → |  | .505 | | ([35](https://www.av8n.com/physics/uncertainty.htm#eq-crank-3-linear)) |

[Equation 35](https://www.av8n.com/physics/uncertainty.htm#eq-crank-3-linear) tells us that if *x* is distributed according to *x* = 2±.02 then 1/*x* is distributed according to 1/*x* = .5±.005. Equivalently we can say that if *x* = 2±1% then 1/*x* = .5±1%. We remark in passing that the percentage uncertainty (aka the relative uncertainty) is the same for *x* and 1/*x*, which is what we expect provided the uncertainty is small.

The Crank Three Times™ method is a type of “what if” analysis. We can also consider it a simple example of an *iterative numerical* method of estimating the uncertainty.

The Crank Three Times™ method is by no means an exact error analysis. It is an approximation. The nice thing is that you can understand the nature of the approximation, and you can see that better and better results are readily available (for a modest price).

One of the glories of the Crank Three Times™ method is that in cases where it doesn’t work, it will tell you it isn’t working, provided you listen to what it’s trying to tell you. If you get asymmetrical error bars, you need to investigate further. Something bad is happening, and you need to check closely to see whether it is a little bit bad or very, very bad.

As far as I can tell, for every flaw that this method has, the sig-figs method has the same flaw plus others ... which means Crank Three Times™ is Pareto superior.

This method requires no new software, no learning curve, and no new concepts beyond the concept of uncertainty itself. In particular, unlike significant digits, it introduces no wrong concepts.

Crank Three Times™ shouldn’t require more than a few minutes of labor. Once a problem is set up, turning the crank should take only a couple of minutes; if it takes longer than that you should have been doing it on a spreadsheet all along. And if you are using a spreadsheet, Crank Three Times™ is super-easy and super-quick.

If you have *N* variables that are (or might be) making a significant contribution to the uncertainty of the result, the Crank Three Times™ method could more precisely be called the Crank 2*N*+1 Times™ method. Here’s the procedure: Set up the spreadsheet and wiggle each variable in turn, and see what happens. Wiggle them *one* at a time, leaving the other *N*−1 at their original, nominal values.

If you are worried about what happens when two of the input variables are simultaneously at the ends of their error bars, you can check that case if you want. However, beware that if there are many variables, checking all the possibilities is exponentially laborious. Furthermore, it is improbable that many variables would simultaneously take on extreme values, and checking extreme cases can lead you to overestimate the uncertainty. For these reasons, and others, if you have numerous variables and need to study the system properly, at some point you need to give up on the Crank Three Times™ method and do a full-blown Monte Carlo analysis.

In the rare situation where you want a worst-case analysis, you can move each variable to whichever end of its error bar makes a positive contribution to the final answer, and then flip them all so that each one makes a negative contribution. In most cases, however, a worst-case analysis is wildly over-pessimistic, especially when there are more than a few uncertain variables.

Remember: there are many cases, especially when there are multiple uncertain variables and/or correlations among the variables and/or nonlinearities, your only reasonable option is Monte Carlo. The Crank Three Times™ method can be considered an ultra-simplified variation of the Monte Carlo method, suitable for introductory reconnaissance.

Here is another example, which is more interesting because it exhibits nonlinearity:

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|  | |  |  |  |  |  |  | | --- | --- | --- | --- | --- | --- | | *x* |  |  |  |  | 1/*x* | | === |  |  |  |  | === | | 2.9 | (high case) |  | → |  | .34 | | 2 | (nominal case) |  | → |  | .5 | | 1.1 | (low case) |  | → |  | .91 | | ([36](https://www.av8n.com/physics/uncertainty.htm#eq-crank-3-nonlinear)) |

[Equation 36](https://www.av8n.com/physics/uncertainty.htm#eq-crank-3-nonlinear) tells us that if *x* is distributed according to *x* = 2±.9 then 1/*x* is distributed according to 1/*x* = .5(+.41−.16). Equivalently we can say that if *x* = 2±45% then 1/*x* = .5(+82%−31%). Even though the error bars on *x* are symmetric, the error bars on 1/*x* are markedly lopsided.

Lopsided error bars are fairly common in practice. Sometimes they are merely a symptom of a harmless nonlinearity, but sometimes they are a symptom of something much worse, such as a singularity or a branch cut in the calculation you are doing.

Here is yet another example, which is interesting because it shows how to handle *correlated* uncertainties in simple cases. The task is to calculate the molar mass of natural bromine, given the nuclide mass for each isotope, and the corresponding natural abundance.

The trick here is to realize that the abundances must add up to 100%. So if one isotope is at the low end of its error bar, the other isotope must be at the high end of its error bar. So the abundance numbers are *anticorrelated*.

(The uncertainties in the mass of each nuclide are negligible.)

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|  |  | nuclide mass |  |  |  | natural |  |  |  | light case |  | nominal case |  | heavy case |  |
|  |  | / Dalton |  |  |  | abundance |  |  |  |  |  |  |  |  |  |
| 79Br |  | 78.9183376(20) |  | × |  | 50.686+.026% |  | = |  | 40.02107 |  |  |  |  |  | more |
| 79Br |  | 78.9183376(20) |  | × |  | 50.686% |  | = |  |  |  | 40.00055 |  |  |  | nominal |
| 79Br |  | 78.9183376(20) |  | × |  | 50.686-.026% |  | = |  |  |  |  |  | 39.98003 |  | less |
| 81Br |  | 80.9162911(30) |  | × |  | 49.314+.026% |  | = |  |  |  |  |  | 39.92410 |  | more |
| 81Br |  | 80.9162911(30) |  | × |  | 49.314% |  | = |  |  |  | 39.90306 |  |  |  | nominal |
| 81Br |  | 80.9162911(30) |  | × |  | 49.314-.026% |  | = |  | 39.88202 |  |  |  |  |  | less |
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|  |  |  |  |  |  |  |  |  |  | 79.90309 |  | 79.90361 |  | 79.90412 |  |  |

So by comparing the three columns (light case, nominal case, and heavy case), we find the bottom-line answer: The computed molar mass of natural bromine is 79.90361(52). This is the right answer based on a particular sample of natural bromine. The usual “textbook” value is usually quoted as 79.904(1), which has nearly twice as much uncertainty, in order to account for sample-to-sample variability.

Note that if you tried to carry out this calculation using “significant figures” you would get the uncertainty wrong. Spectacularly wrong. Off by two orders of magnitude. The relative uncertainty in the molar mass is two orders of magnitude smaller than the relative uncertainty in the abundances.